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ON RIESZ SUBSETS OF ABELIAN DISCRETE GROUPS

BY

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ABSTRACT

We study the class of the Riesz subsets of abelian discrete groups, that is, the sets for which the F. and M. Riesz theorem extends. We show that the "classical" tools of the theory — Riesz projections, localization in the Bohr sense, products — are leading to Riesz sets which are satisfying nice additional properties, e.g., the Mooney–Havin result extends to this class. We give an alternative proof of a result of A. B. Alexandrov, and we improve a construction of H. P. Rosenthal. The connection is made between this class and the M-structure theory. We show a result of convergence at the boundary for holomorphic functions on the polydisc. The Bourgain–Davis result on convergence of analytic martingales is improved.

0. Introduction. Notations

The classical F. and M. Riesz theorem (see [28], p. 335) asserts that a measure μ on the unit circle $\mathbf{T}$, such that

$$\int_{\mathbf{T}} e^{-int} d\mu(t) = 0$$

for every negative integer n, is absolutely continuous with respect to Lebesgue measure. It is natural to seek for extensions of this result within the frame of harmonic analysis on groups. Let me recall that a subset Λ of an abelian discrete group Γ is a *Riesz set* if every Radon measure μ on $G = \hat{\Gamma}$, such that $\hat{\mu}(\alpha) = 0$ for every $\alpha \notin \Lambda$, is absolutely continuous with respect to the Haar measure of G. With this terminology, the F. and M. Riesz theorem asserts that $\mathbf{N}$ is a Riesz subset of $\mathbf{Z}$.

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The goal of this work is to investigate the class of Riesz subsets of abelian discrete groups. The natural frame for this research is the study of Lacunary Fourier series, or equivalently, of the "thin" subsets of abelian discrete groups. The bibliography of Riesz sets should include — at least — the references [1], [5], [20], [21], [25], [27], [31], [32]. But I think it can be said that this property is far from being well understood (see Remark 4.7).

Let us summarize the contents of this paper. The main problem is clearly to have a control on the Fourier transform of the singular part of a given Radon measure. Motivated by the classical differentiation technique, we introduce the notion of *nicely placed subset* of an abelian discrete group (Definition 1.4). The corresponding class of "small" sets (see 1.1) is the family of Shapiro sets (Definition 1.8). We show (Theorem 1.9) that Shapiro sets are Riesz sets, for which the Mooney–Havin theorem ([14], [23]) extends.

In §2 we use several techniques — Riesz projections in ordered groups, localization in the Bohr compactification sense, stability under product — for showing how to construct nicely placed or Shapiro sets.

Then §3 is devoted to examples. We show that all the previous examples of Riesz sets are actually Shapiro; however, an example due to A. B. Alexandrov (3.8) shows that the two classes do not coincide. We give (3.1) a proof of the Mooney–Havin result ([14], [23]) which uses only the very basic properties of $H^1(D)$. An alternative proof of a result of A. B. Alexandrov ([1], Appendix) is given (3.5). We focus in (3.6) on arithmetical examples which tend to show that, even in $\mathbf{Z}$, the situation is excitingly complicated. A construction of H. P. Rosenthal [25] is improved in (3.8.1).

In §4 we consider some applications of the previous results. Proposition (4.1) makes the connection between the Shapiro sets and the M-structure theory in Banach spaces. We show a result of convergence at the boundary for holomorphic functions of the polydisc (4.3). With (4.1) and the Hankel operators, we make a connection (4.5) between the decompositions of functions in L^1_S — S semi-group — and the Riesz and Shapiro properties. We prove and show how to improve in (4.6) the result ([8], Cor. 4.3) of convergence of analytic martingales.

Notations

Γ denotes an abelian discrete group, and $G = \hat{\Gamma}$ its compact dual group. The Haar probability on G is denoted by m. We denote by $M(G)$ the space of Radon measures on G. If μ is a Radon measure, $\hat{\mu}$ denotes its Fourier transform; for any subset Λ of Γ, we let $M_\Lambda(G)$ be the space of μ in $M(G)$ such that $\hat{\mu}(\alpha)=0$ for $\alpha \notin \Lambda$. The spaces $L^1_\Lambda(G)$, $\mathscr{C}_\Lambda(G)$, $L^\infty_\Lambda(G)$ are defined in a similar way. The notation $[\Lambda]$ (defined after 1.4) denotes the smallest nicely placed set which contains Λ; the notation $\tilde{\Lambda}$ is defined before 1.7. We denote by μ_a — resp. μ_s — the absolutely continuous — resp. singular — part of a Radon measure μ with respect to Haar measure. A subset Λ of Γ is a $\Lambda(1)$ set [27] if there exists $q<1$ such that $\| P \|_1 \leqq K . \| P \| q$ for any trigonometric polynomial P supported by Λ. The set Λ is $\Lambda(1)$ if L^1_Λ is reflexive ([2], [26]). The L^0-metric is defined by

$$d(f,g)=\int |f-g|(1+|f-g|)^{-1}dm$$

and the $L(1,\infty)$ ("weak L^1") quasi-norm by

$$\| f \|_{1,\infty}=\sup_{\lambda \geqq 0}\{\lambda P(|f| \geqq \lambda)\}.$$

The notation $B_1(X)$ denotes the closed unit ball of a Banach space X. The other notations we use are classical or will be recalled in the article.

1. Nicely placed sets and Shapiro sets

Our first lemma is a fairly general principle for showing that certain sets are Riesz. If E is a set and $\mathscr{C} \subset P(E)$ is a family of subsets of E, we denote by $\dot{\mathscr{C}}$ the biggest hereditary class contained in $\mathscr{C}$; that is,

$$\dot{\mathscr{C}}=\{A \in P(E) \mid \forall B \subset A, B \in \mathscr{C}\}.$$

The family $\dot{\mathscr{C}}$ is the "family of small sets" which naturally corresponds to the family $\mathscr{C}$. With this terminology, the following holds:

LEMMA 1.1. *Let Γ be an abelian discrete group, and $\mathscr{C} \subseteq P(\Gamma)$ a family of subsets of Γ. If every $\Lambda \in \mathscr{C}$ satisfies:*

$$(1) \qquad \forall \mu \in M_\Lambda(G), \quad \mu_s \in M_\Lambda(G)$$

then every $\Lambda \in \dot{\mathscr{C}}$ is a Riesz set.

PROOF. Let Λ be in $\dot{\mathscr{C}}$, and $\mu \in M_\Lambda(G)$. Since $\dot{\mathscr{C}} \subseteq \mathscr{C}$, we have by (1) that $\mu_s \in M_\Lambda(G)$. Let us assume that $\mu_s \neq 0$; then there exists $\alpha \in \Lambda$ such that $\hat{\mu}_s(\alpha) \neq 0$. We let $\Lambda'=\Lambda \setminus \{\alpha\}$; Λ' is in $\mathscr{C}$ since $\Lambda \in \dot{\mathscr{C}}$. Consider $\mu'=\mu-\hat{\mu}(\alpha)\alpha$. It is clear that $\mu' \in M_{\Lambda'}(G)$ and thus by (1), $(\mu')_s \in M_{\Lambda'}(G)$. But $(\mu')_s=\mu_s$ and thus $\mu_s \in M_{\Lambda'}(G)$ and $\hat{\mu}_s(\alpha)=0$; this is a contradiction. $\square$

EXAMPLES. If the class $\mathscr{C}_0$ is the class of all the subsets Λ of Γ which satisfy (1), then $\mathscr{C}_0$ is exactly the class of the Riesz subsets of Γ. In [21], Y. Meyer considered the class $\mathscr{C}$ of the subsets of Γ which are closed for the topology of pointwise convergence on $\hat{\Gamma}$. In [31], J. Shapiro considered the class $\mathscr{C}$ of the subsets Λ satisfying: there exists $0 < p < 1$ such that L^1_Λ is relatively closed in L^1 for the quasi-norm $\| \ \|_p$. In this paper we will define (Definition 1.4) and study the class of the "nicely placed" sets, which contains Y. Meyer's and J. Shapiro's classes.

I am indebted to N. J. Kalton for the next lemma, which is a slight improvement of a classical result of J. Komlos.

LEMMA 1.2. *Let $(f_n)_{n \geqq 1}$ be a bounded sequence in $L^1(\Omega, P)$. There exists a subsequence $(\tilde{f}_k)_{k \geqq 1}$ of (f_n) such that the sequence*

$$g_n = \frac{1}{n}\left(\sum_{k=1}^{n} \tilde{f}_k\right)$$

converges almost everywhere and in $L(1, \infty)$. Moreover, the conclusion remains true for any subsequence of $(\tilde{f}_k)$.

PROOF. By Komlos's theorem ([18]), there exists a subsequence $(f'_k)_{k \geqq 1}$ of (f_n) such that the Cesaro means of any subsequence of $(f'_k)_{k \geqq 1}$ converge a.e.

Let us apply the subsequence splitting lemma to (f'_k); there exists a subsequence $(f''_k)_{k \geqq 1}$ of (f'_k) which can be written

$$f''_k = g_k + j_k + h_k$$

where the sequence $(g_k)_{k \geqq 1}$ is equi-integrable, the sequence $(j_k)_{k \geqq 1}$ satisfies $\| j_k \|_1 < 2^{-k}$ for every k, and the functions (h_k) are mutually disjoint. By modifying (j_k) if necessary, we may assume that $\| h_k \|_\infty < \infty$ for every k.

Since the sequence (h_k) is tending to 0 a.e. and the sequence (g_k) is equi-integrable, any subsequence of (g_k) will converge in Cesaro mean a.e. and in $\| \ \|_1$. Therefore it is enough to show that $(h_k)_{k \geqq 1}$ has a subsequence which converges to 0 in Cesaro mean in $L(1, \infty)$, as well as its subsequences. If $(h_k)_{k \geqq 1}$ is uniformly bounded, this is clearly true; therefore, we may assume without loss of generality that $\| h_{k+1} \|_\infty \geqq 1 + \| h_k \|_\infty$ for every k.

Let $h'_1 = h_1$ be given. We construct the subsequence $(h'_k)_{k \geqq 1}$ in such a way that

$$\forall k, \quad P(|h'_{k+1}| > 0) \, . \, \| h'_k \|_\infty \leqq 2^{-k}. \tag{2}$$

This is possible since the (h_n) are disjoint and thus $\lim_{n\to\infty} P(|h_n|>0)=0$. Since the (h'_n) are disjoint, we have

$$\forall x \geqq 0, \quad P\left(\sum_{i=1}^{n} |h'_i| \geqq x\right) = \sum_{i=1}^{n} P(|h'_i| \geqq x).$$

For a given x, let i_0 be such that $\| h'_{i_0+1} \|_\infty \geqq x \geqq \| h'_{i_0} \|_\infty$. We have for any n

$$xP\left(\sum_{i=1}^{n} |h'_i| \geqq x\right) = x\sum_{i=1}^{n} P(|h'_i| \geqq x) = x\sum_{i_0+1}^{n} P(|h'_i| \geqq x)$$

and thus

$$xP\left(\sum_{i=1}^{n} |h'_i| \geqq x\right) \leqq xP(|h'_{i_0+1}| \geqq x) + \| h'_{i_0+1} \|_\infty \sum_{i_0+2}^{n} P(|h'_i| \geqq x)$$

and by (2)

$$xP\left(\sum_{i=1}^{n} |h'_i| \geqq x\right) \leqq \| h'_{i_0+1} \|_1 + \sum_{k=i_0+2}^{n} 2^{-k} \leqq 1 + \| h'_{i_0+1} \|_1.$$

If M is such that $\| h'_k \|_1 \leqq M$ for every k, we have

$$\forall x \geqq 0, \quad \forall n, \quad xP\left(\sum_{i=1}^{n} |h'_i| \geqq x\right) \leqq M+1. \tag{3}$$

Applying (3) to $x = nx'$, we get

$$\forall x' \geqq 0, \quad \forall n, \quad x'P\left(\frac{1}{n}\sum_{i=1}^{n} |h'_i| \geqq x'\right) \leqq (M+1)/n$$

and this proves that $(1/n)(\Sigma_{i=1}^{n} h'_i)$ converges to zero in $L(1,\infty)$. Moreover, any subsequence (h''_i) of (h'_i) satisfies also (2) and the proof shows that the conclusion remains true for (h''_i). □

Let us prove now:

Lemma 1.3. *Let C be a convex subset of $L^1(\Omega, P)$, closed and bounded in $\| \ \|_1$. The following are equivalent*:

(1) *C is closed in L^0.*

(2) *C is closed in $L(1,\infty)$.*

(3) *C is the image of its closure $\overline{C}^*$ in (L^{1**}, w^*) under the canonical projection π from L^{1**} to L^1.*

Proof. (1)⇒(2) is obvious.

(2)$\Rightarrow$(1). Let $(f_n)_{n\geq 1}$ be a sequence in C converging to f in L^0. There exists a subsequence (f'_n) which tends to f a.e.; by Lemma 1.2, there exists a subsequence (f''_n) of (f'_n) which converges a.e. and in $L(1,\infty)$ in Cesaro mean. But $\lim_{n\to\infty} f''_n = f$ a.e. and therefore

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n} f''_k\right)=f \qquad \text{a.e. and in } L(1,\infty)$$

and this implies $f\in C$ since C is convex and $L(1,\infty)$ closed.

(1)$\Leftrightarrow$(3) is the main result of [10]. □

It is now time to adapt ([12], Def. 2):

DEFINITION 1.4. Let X be a closed subspace of L^1. The space X is said to be *nicely placed* if the unit ball $B_1(X)$ of X is closed in $L(1,\infty)$. A subset Λ of an abelian discrete group Γ is nicely placed if the space $L^1_\Lambda(\hat{\Gamma})$ is nicely placed in $L^1(\hat{\Gamma})$.

If Y is now any subspace of L^1, the family $\mathcal{Y}$ of the nicely placed subspaces of L^1 containing Y is clearly stable by intersection; we note $[Y]=\bigcap\mathcal{Y}$ the smallest nicely placed subspace containing Y. If now $Y=L^1_\Lambda$ is translation invariant, it is clear that $[Y]$ is also translation invariant. Let $[\Lambda]$ be the subset of Λ which satisfies $[L^1_\Lambda]=L^1_{[\Lambda]}$. The set $[\Lambda]$ is clearly the smallest nicely placed subset of Γ containing Λ.

Our next lemma will show that the family $\mathcal{C}$ of the nicely placed sets Λ satisfies the assumption (1) of Lemma 1.1.

LEMMA 1.5. *Let G be a compact abelian group, Λ a subset of $\Gamma=\hat{G}$ and $\mu\in M_\Lambda(G)$. If μ_s denotes the singular part of μ, one has $\mu_s\in M_{[\Lambda]}(G)$.*

PROOF. Let F be the filter of neighborhoods of 0 in G, and $V\in F$; we let

$$k_v = m(V)^{-1}\,.\,\mathbf{1}_v.$$

By the differentiation lemma of [6], one has

$$\lim_F \|\mu_a * k_v - \mu_a\|_1 = 0, \tag{4}$$

$$\lim_F \|\mu_s * k_v\|_p = 0 \qquad \forall\, 0<p<1. \tag{5}$$

This shows that μ_a belongs to the closure of the set $C=\mu * B_1(L^1)$ in $\|\ \|_p$ ($p<1$). By Lemma 1.3, μ_a belongs to the closure of C in $L(1,\infty)$. But C is a bounded subset of $L^1_\Lambda(G)$ since $\mu \in M_\Lambda(G)$ and therefore $\mu_a \in L^1_{[\Lambda]}(G)$. Since $\mu_s = \mu - \mu_a$ and $\Lambda \subset [\Lambda]$, the proof is complete. □

Lemma 1.1 leads to considering the hereditary class $\mathscr{C}$ corresponding to the class of nicely placed sets. Our terminology is motivated by the use we are making of the ideas of [31].

Definition 1.6. Let Λ be a subset of the abelian discrete group Γ. Λ is a *Shapiro set* if every subset Λ' of Λ is nicely placed.

Our next lemma is an useful test for proving that a set is Shapiro. For any subset Λ of Γ, we denote by $\tilde{\Lambda}$ the set of α's in Λ such that the restriction of $F_\alpha = f \to \hat{f}(\alpha)$ to $B_1(L^1_\Lambda)$ is $L(1,\infty)$-continuous. With this notation, one has:

Lemma 1.7. *Let Λ be a nicely placed subset of Γ. The following are equivalent*:

(1) Λ *is Shapiro*.

(2) $\Lambda = \tilde{\Lambda}$.

Proof. (2)$\Rightarrow$(1). Let Λ' be a subset of Λ. One has

$$L^1_{\Lambda'} = \bigcap_{\alpha \in \Lambda \setminus \Lambda'} \{ f \in L^1_\Lambda \mid \hat{f}(\alpha) = 0 \} \tag{6}$$

and since $[\Lambda] = \Lambda = \tilde{\Lambda}$, (6) shows that Λ' is nicely placed.

(1)$\Rightarrow$(2). Let α be in Λ. We have to show that F_α is $L(1,\infty)$-continuous on $B_1(L^1_\Lambda)$. By assumption, the set $\Lambda' = \Lambda \setminus \{\alpha\}$ is nicely placed. If F_α is not $L(1,\infty)$ continuous on $B_1(L^1_\Lambda)$, there exists a sequence $(f_n)_{n\geq 1}$ in $B_1(L^1_\Lambda)$ such that $(f_n)_{n\geq 1}$ tends to 0 in $L(1,\infty)$ and $\lim_{n\to\infty} \hat{f}_n(\alpha) = t \neq 0$. Consider $g_n = f_n - \hat{f}_n(\alpha) \,.\, \alpha$; (g_n) is a bounded sequence in $L^1_{\Lambda'}$, which tends to $(-t) \,.\, \alpha$ in $L(1,\infty)$; this is impossible since Λ' is nicely placed. □

Now let us connect the nicely placed sets with the properties of L^1/L^1_Λ:

Lemma 1.8. *Let Λ be a nicely placed subset of Γ. Then the space $L^1/L^1_\Lambda(\hat{\Gamma})$ is weakly sequentially complete.*

Proof. This lemma is a special case of ([12], Th. 3). Let us prove it for completeness. Since Λ is nicely placed, we have by Lemma 1.3

$$B_1(L^1_\Lambda) = \pi(\overline{B_1(L^1_\Lambda)^*})$$

and thus

$$L^{1\perp\perp}_\Lambda = L^1_\Lambda \oplus (L^{1\perp\perp}_\Lambda \cap \operatorname{Ker} \pi).$$

This implies that

$$(L^1/L^1_\Lambda)^{**} = L^{1**}/L^{1\perp\perp}_\Lambda = L/L^1_\Lambda \oplus \operatorname{Ker}\pi/(\operatorname{Ker}\pi \cap L^{1\perp\perp}_\Lambda).$$

Moreover, if $x \in (L^1/L^1_\Lambda)^{**}$ and $x = t + u$ in the above decomposition, an easy computation shows that $\| x \| = \| t \| + \| u \|$. For showing that L^1/L^1_Λ is weakly sequentially complete, it is therefore enough to show that if Y is a Banach space and $u \in Y^{**} \setminus \{0\}$ satisfies

$$\forall t \in Y, \quad \| u + t \| = \| u \| + \| t \| \tag{7}$$

then u cannot be of the first Baire class.

We consider u as a function on the compact space $(B_1(Y^*), w^*)$. Let $\hat{u}$ be the u.s.c. hull of u; it is easily checked that u is a concave function, and the Hahn–Banach theorem tells us that

$$\hat{u} = \inf\{ y \mid y \text{ finite w*-continuous}, y \geqq u \text{ on } B_1(Y^*)\}.$$

Such a y has the form $y = t + \lambda$, for some $t \in Y$ and $\lambda \in R$. We want

$$t + \lambda \geqq u \quad \text{on } B_1(Y^*),$$

that is,

$$\lambda \geqq u - t \quad \text{on } B_1(Y^*),$$

It is enough to take $\lambda = \| u - t \|$. Thus we have

$$\forall x \in B_1(Y^*), \quad \hat{u}(x) = \inf\{t(x) + \| u - t \| \mid t \in Y\}$$

and by (7),

$$\hat{u}(x) = \inf\{t(x) + \| u \| + \| t \| \mid t \in Y\},$$

but $\| t \| \geqq |t(x)|$ for every $t \in Y$ and thus $\hat{u} \geqq \| u \|$. Since the reverse inequality is trivial, $\hat{u} = \| u \|$. Finally, if $\check{u}$ is the l.s.c. hull of u on $(B_1(Y^*), w^*)$, we clearly have $\check{u}(t) = -\hat{u}(-t)$ and thus $\check{u} = -\| u \|$. Since $\| u \| > 0$, this shows that the restriction of u to the compact space $(B_1(Y^*), w^*)$ has no points of continuity. By Baire's theorem, this implies that u is not of the first Baire class. □

We are now ready to prove the main result of this section

Theorem 1.9. *Let Λ be a Shapiro subset of an abelian discrete group Γ. Then Λ is Riesz, and the space L^1/L^1_Λ is weakly sequentially complete.*

Proof. By Lemmas 1.1 and 1.5, Λ is Riesz. By Lemma 1.8, L^1/L^1_Λ is weakly sequentially complete. □

REMARKS. (1) In the results of this section, we may replace the topology $L(1, \infty)$ by L^p $(0 \leqq p < 1)$ or by the topologies of the Lorentz spaces $L(1, p)$ (see [17] for a definition); these last topologies are actually finer than $L(1, \infty)$ and thus the corresponding results are, at least formally, better. More generally, we can use every topology of vector lattice X such that $L^1 \subset X \subset L^0$ and "Lemma 1.2" is true. It seems natural, however, to formulate the results in the finest "classical" topology of that kind, namely, in the $L(1, \infty)$ topology.

(2) Let us translate the properties we are considering in Theorem 1.9 into existence results. It is easily seen, by the Bishop–Rudin–Carleson theorem [4], that Λ is a Riesz set if and only if:

$\forall K$ compact in G with $m(K) = 0$, $\forall U$ open containing K, $\forall \varepsilon > 0$, $\forall f \in \mathscr{C}(K)$, there exists $g \in \mathscr{C}(G)$ such that:

(1) $\| g \|_\infty = \| f \|_\infty$,

(2) $g = f$ on K, $|g| < \varepsilon$ on $G \setminus U$,

(3) $\hat{g}(\alpha) = 0$ for every $\alpha \in \Lambda$.

$L^1/L^1_\Lambda(G)$ is weakly sequentially complete if and only if for every sequence $(f_n)_{n \geqq 1}$ in $L^1(G)$ such that

$$\lim_{n \to \infty} \int g f_n dm \quad \text{exists for every } g \in L^\infty_{\Gamma \setminus \Lambda}(G)$$

there exists $\phi \in L^1(G)$ with

$$\lim_{n \to \infty} \int g \phi dm = \lim_{n \to \infty} \int g f_n dm, \qquad \forall g \in L^\infty_{\Gamma \setminus \Lambda}(G).$$

2. Techniques of construction of nicely placed and Shapiro sets

Our first technique is dealing with order structures on discrete groups. Let us recall that a totally ordered group is a discrete group equipped with a total order compatible with the group structure. Our main reference for this notion is ([29], Chap. 8). Let us prove:

THEOREM 2.1. *Let Γ be a totally ordered discrete group, and Λ a subset of Γ such that*

$$(*) \qquad \Lambda \cap \{\alpha' \leqq \alpha\} \text{ is } \Lambda(1) \text{ for every } \alpha \in \Gamma.$$

Then Λ is a Shapiro subset of Γ.

PROOF. Let Γ^+ be the set of positive elements of Γ.

Since the property $(*)$ is clearly hereditary, it is enough to prove that Λ is nicely placed. Let $(f_n)_{n\geqq 1}$ be a sequence in $B_1(L^1_\Lambda)$, which converges to f in $L(1,\infty)$, and let α_0 be in $\Gamma\setminus\Lambda$; we have to prove that $\hat{f}(\alpha_0)=0$.

Let $\alpha\in\Gamma$ be such that $\alpha>\alpha_0$. For any trigonometric polynomial $P=\Sigma_\gamma a_\gamma\gamma$ on G, we define the Riesz projection R^-_α by

$$R^-_\alpha(P)=R^-_\alpha\left(\sum_\gamma a_\gamma\gamma\right)=\sum_{\gamma<\alpha}a_\gamma\gamma.$$

By ([29], Th. 8.7.6), R^-_α is continuous from L^1 to $L(1,\infty)$. We let $\Gamma^+_\alpha=\{\alpha'\geqq\alpha\}$. If $\hat{P}$ is supported by Λ, we have $R^-_\alpha(P)\in L^1_{\Lambda\setminus\Gamma^+_\alpha}$. Since $(\Lambda\setminus\Gamma^+_\alpha)$ is $\Lambda(1)$, the $\|\ \|_1$ and $L(1,\infty)$ topologies coincide on L^1_Λ [16]. Therefore R^-_α is $\|\ \|_1$-continuous from L^1_Λ into $L^1_{\Lambda\setminus\Gamma^+_\alpha}$, and thus we have

$$L^1_\Lambda=L^1_{\Lambda\setminus\Gamma^+_\alpha}\oplus L^1_{\Lambda\cap\Gamma^+_\alpha}.$$

Let (f'_n) be a subsequence of (f_n) which converges a.e.; we let $g'_n=R^-_\alpha(f'_n)$ and $h'_n=f'_n-g'_n$. The sequences (g'_n) and (h'_n) are $\|\ \|_1$ bounded and thus by Lemma 1.2 there exist subsequences (g''_n) and (h''_n), indexed by the same set, which converge in Cesaro mean a.e. and in $L(1,\infty)$ to g and h, respectively. We have $f=g+h$.

Since $(\Lambda\setminus\Gamma^+_\alpha)$ is $\Lambda(1)$, the space $L^1_{\Lambda\setminus\Gamma^+_\alpha}$ is $L(1,\infty)$ closed and thus $g\in L^1_{\Lambda\setminus\Gamma^+_\alpha}$ and $\hat{g}(\alpha_0)=0$. On the other hand, the algebra $\mathscr{C}_{\Gamma^+}(G)$ is a Dirichlet algebra [15], and this implies by ([9], p. 217) and Lemma 1.3 that $L^1_{\Gamma^+}(G)$ is nicely placed in $L_1(G)$; therefore, Γ^+ and Γ^+_α are nicely placed subsets of Γ. This shows that $h\in L^1_{\Gamma^+_\alpha}(G)$ and thus $\hat{h}(\alpha_0)=0$ and $\hat{f}(\alpha_0)=\hat{g}(\alpha_0)+\hat{h}(\alpha_0)=0$. □

We may apply this result with the help of the Lie algebra of a compact group G. Let us prove the easy Corollary 2.2. In the next statement, **R** is equipped with its natural order and the discrete topology.

Corollary 2.2. *Let G be a compact group, and Λ a subset of $\hat{G}$. If there exists a dense one parameter subgroup $\psi(\mathbf{R})$ of G such that the subset $\psi^*(\alpha)=\{\alpha\circ\psi \mid \alpha\in\Lambda\}$ of* **R** *satisfies the condition* (*) *of Theorem* 2.1, *then Λ is a Shapiro subset of $\hat{G}$.*

Proof. Let us notice first that any subgroup Γ' of an abelian discrete group Γ is nicely placed in Γ. Indeed, let $\Gamma'^\perp$ be the subgroup of $G=\hat{\Gamma}$ defined by

$$\Gamma'^\perp=\{g\in G \mid \gamma'(g)=1\ \ \forall\gamma'\in\Gamma'\}.$$

It is clear that

$$L^1_{\Gamma'}(G)=\{f\in L^1(G) \mid f\circ\tau_g=f\ \forall g\in\Gamma'^\perp\}$$

where $\tau_g(u) = gu$ for $u \in G$. But the condition $f \circ \tau_g = f$ is clearly stable under $L(1, \infty)$ limits and thus $L^1_{\Gamma'}$ is $L(1, \infty)$ closed in $L^1(G)$.

Now it is clear that if Λ is nicely placed — resp. Shapiro — in $\Gamma' \subset \Gamma$, then Λ is nicely placed — resp. Shapiro — in Γ. Applying this remark to $\Gamma' = \psi(\hat{G})$ and $\Gamma = \mathbf{R}$ proves Corollary 2.2. □

Corollary 2.2 takes care of the Archimedean applications. We will see below (3.5, 3.7 and 4.6) that the non-Archimedean case could also be of interest.

The second technique we will use is a technique of "localization". If Γ is a discrete abelian group, we denote by τ the topology induced on Γ by the pointwise convergence on $G = \hat{\Gamma}$. This topology is induced on Γ by the Bohr compactification $b\Gamma$ of Γ. The ideas we will use here appeared first in [21].

If $\mathscr{C} \subset P(\Gamma)$ is a family of subsets of Γ, we will say that $\mathscr{C}$ is *localizable* if the following holds:

$$\Lambda \in \mathscr{C} \Leftrightarrow \forall \alpha \in \Gamma,\ \exists\, V_\alpha \text{ a } \tau\text{-neighborhood of } \alpha \text{ in } \Gamma \text{ such that } (\Lambda \cap V_\alpha) \in \mathscr{C}.$$

REMARK. It is easy to check that if $\mathscr{C}$ is localizable, then the hereditary subfamily $\acute{\mathscr{C}}$ of $\mathscr{C}$ (see 1.1) is localizable.

With this terminology, the following holds:

THEOREM 2.3. *The following classes are localizable*:
(1) $\mathscr{C}_0 = \{\Lambda \subseteq \Gamma \mid \mu \in M_\Lambda(G) \Rightarrow \mu_s \in M_\Lambda(G)\}$,
(2) *the class of nicely placed sets*,
(3) *the class of Riesz sets*,
(4) *the class of Shapiro sets*.

PROOF. By the above remark, it is enough to prove (1) and (2).

(1) Let $\Lambda \subseteq \Gamma$ be such that

$$\forall \alpha \in \Gamma, \quad \exists\, V_\alpha \quad \text{s.t.} \quad (V_\alpha \cap \Lambda) \in \mathscr{C}_0. \tag{8}$$

The algebra $A(\Gamma)$ is regular and therefore there exists a discrete measure ν on $G = \hat{\Gamma}$ such that

$$\begin{cases} \hat{\nu}(\alpha) = 1, \\ \hat{\nu}(\lambda) = 0 \qquad \forall \lambda \in \Gamma \setminus V_\alpha. \end{cases} \tag{9}$$

Let μ be in $M_\Lambda(G)$, and $\alpha \notin \Lambda$; we need to prove that $\hat{\mu}_s(\alpha) = 0$. Let V_α be a

τ-neighborhood like in (8), and ν like in (9); we have $\mu * \nu \in M_{\Lambda \cap V_\alpha}$ and $(\mu * \nu)_s = \mu_s * \nu$ since ν is discrete.

$(\Lambda \cap V_\alpha)$ belongs to $\mathscr{C}_0$ and thus $(\mu * \nu)_s = \mu_s * \nu$ belongs to $M_{\Lambda \cap V_\alpha}$; since $\alpha \notin \Lambda$, we have

$$\widehat{\mu_s * \nu}(\alpha) = \hat{\mu}_s(\alpha) = 0.$$

Let us note that, up to the terminology, the above proof is taken from [21].

(2) For this part of the proof we need:

LEMMA 2.4. *Let ν be a discrete measure, and $(f_n)_{n \geq 1}$ a bounded sequence in $L^1(G)$ which converges to f in $L(1, \infty)$. We let $C_\nu(f) = f * \nu$; then $\lim_{n \to \infty} C_\nu(f_n) = C_\nu(f)$ in $L(1, \infty)$.*

PROOF. Let $\varepsilon > 0$ be given. We assume that $\| f_n \|_1 \leqq M$ for every n. There exists a measure ν' with finite support such that $\| \nu' - \nu \|_1 \leqq M^{-1} \cdot \varepsilon / 3$. We have, for $\| g \|_1 \leqq M$,

$$\| C_\nu(g) - C_{\nu'}(g) \|_1 \leqq \| \nu - \nu' \|_1 \, \| g \|_1 \leqq \varepsilon / 3. \tag{10}$$

Since ν' is finitely supported, it is easily seen that

$$\exists N_0 \text{ s.t. } \forall n \geqq N_0, \quad \| c_{\nu'}(f_n) - c_{\nu'}(f) \|_{1,\infty} \leqq \varepsilon / 3. \tag{11}$$

It is clear that (10) and (11) prove the lemma. □

Our next lemma will be very useful.

LEMMA 2.5. *Let Λ be a subset of Γ and $\alpha \in \Lambda$. If there exists a τ-neighborhood V_α of α such that $\alpha \notin [V_\alpha \cap \Lambda]$, then $\alpha \notin [\Lambda]$.*

PROOF. Let ν be a discreted measure which satisfies (9). Lemma 2.4 shows that if X is nicely placed, then $C_\nu^{-1}(X)$ is nicely placed. We have

$$C_\nu(L^1_\Lambda) \subseteq L^1_{V_\alpha \cap \Lambda}$$

and thus

$$C_\nu(L^1_{[\Lambda]}) \subseteq L^1_{[V_\alpha \cap \Lambda]}. \tag{12}$$

Since $C_\nu(\alpha) = \alpha \notin [V_\alpha \cap \Lambda]$, (12) shows that $\alpha \notin [\Lambda]$. □

Let us conclude the proof of 2.3(2): if $\alpha \in \Gamma \setminus \Lambda$, there exists V_α such that $[V_\alpha \cap \Lambda] = V_\alpha \cap \Lambda$ and thus $\alpha \notin [V_\alpha \cap \Lambda]$; by Lemma 2.5 $\alpha \notin [\Lambda]$. □

Lemma 2.5 actually implies

COROLLARY 2.6. (1) *Let Λ_1 be a nicely placed subset of Γ and Λ_2 a τ-closed subset of Γ. Then $(\Lambda_1 \cup \Lambda_2)$ is nicely placed. In particular, every τ-closed subset is nicely placed.*

(2) *Let Λ_1 be a Riesz and τ-closed subset of Γ, and Λ_2 be Shapiro. Then every set Λ such that $\Lambda_1 \subset \Lambda \subset \Lambda_1 \cup \Lambda_2$ is Riesz and nicely placed.*

PROOF. (1) If $\alpha \in \Gamma \backslash (\Lambda_1 \cup \Lambda_2)$, there exists V_α such that $V_\alpha \cap (\Lambda_1 \cup \Lambda_2) \subset V_\alpha \cap \Lambda_1$ and since $[V_\alpha \cap \Lambda_1] \subset [\Lambda_1] = \Lambda_1$ we have $\alpha \notin [V_\alpha \cap (\Lambda_1 \cup \Lambda_2)]$; by 2.5, $\alpha \notin [\Lambda_1 \cup \Lambda_2]$. The special case is obtained by taking $\Lambda_1 = \varnothing$.

(2) By (1), such a Λ is nicely placed, and it is Riesz by [(21]). □

The localization gives several examples of stability by union. Let us consider now the stability under products:

THEOREM 2.7. *Let Λ and Λ' be nicely placed — resp. Shapiro — subsets of the countable discrete group Γ and Γ'. Then $\Lambda \times \Lambda'$ is a nicely placed — resp. Shapiro — subset of $\Gamma \times \Gamma'$.*

PROOF. First:

LEMMA 2.8. *Let $(f_n)_{n \geqq 1}$ be a sequence in $L^1(\Omega \times \Omega', \mathbf{P} \otimes \mathbf{P}')$. We assume that $\| f_n \|_1 \leqq 1$ and that $\lim_{n\to\infty} f_n = f$ a.e. For $w' \in \Omega'$, we let $\phi_{n,w'}(w) = f_n(w, w')$ and $\phi_{w'}(w) = f(w', w)$. Then for almost every $w' \in \Omega'$, one has:*

$$(13) \qquad \begin{cases} (1)\ \phi_{n,w'} \in L^1(\Omega)\ \forall n, \\ (2)\ \lim_{n\to\infty} \phi_{n,w'} = \phi_{w'}\ a.e., \\ (3)\ \underline{\lim}\ \| \phi_{n,w'} \|_1 < \infty. \end{cases}$$

PROOF. (1) is an immediate consequence of $f_n \in L^1$ for every n. (2) is clear. Let us prove (3).

For each $k \geqq 1$, let

$$A_{n,k} = \{ w' \in \Omega' \mid \| \phi_{n,w'} \|_1 \geqq k \}.$$

Since $\| f_n \|_1 \leqq 1$, we have

$$(14) \qquad \forall n, \quad \mathbf{P}'(A_{n,k}) \leqq k^{-1}.$$

Let us consider now

$$B_k = \{ w' \mid \exists \text{ infinitely many } n \text{ s.t. } w' \notin A_{n,k} \}.$$

It is clear that

$$\Omega' \backslash B_k = \bigcup_j \left(\bigcap_{n \geqq j} A_{n,k} \right).$$

By (14),

$$\mathbf{P}'\left(\bigcap_{n \geqq j} A_{n,k} \right) \leqq k^{-1} \quad \text{for every } j.$$

The union over j is increasing, and therefore

$$\forall k, \quad \mathbf{P}'(\Omega' \backslash B_k) \leqq k^{-1}$$

and thus $(\Omega' \backslash \bigcup_{k \geqq 1} B_k)$ is a negligible set. If w' does not belong to this set, the condition (3) is clearly satisfied. □

Let us prove that $\Lambda \times \Lambda'$ is nicely placed if Λ and Λ' are.

Let $(f_n)_{n \geqq 1}$ be a sequence in the unit ball of $L^1_{\Lambda \times \Lambda'}$, which converges to f in $L(1, \infty)$. Up to a subsequence, we may assume that (f_n) converges to f a.e. For $\gamma \notin \Lambda$, we have $\hat{f}_n(\gamma, \gamma') = 0$ for every $\gamma' \in \Gamma'$, and thus

$$(15) \qquad z' \text{ a.e.}, \quad \int_G \gamma(-z) f_n(z, z') dm(z) = 0$$

for every n. Since $\Gamma \backslash \Lambda$ is countable, (15) implies

$$(16) \qquad z' \text{ a.e.} \quad \int_G \gamma(-z) f_n(z, z') dm(z) = 0 \quad \forall n, \quad \forall \gamma \notin \Lambda$$

and (16) means that z' a.e., the function

$$\phi_{n,z'}(z) = f_n(z, z')$$

belongs to L^1_Λ for every n. Since Λ is nicely placed, we have by Lemma 2.8 that $\phi_{z'}(z) = f(z, z')$ belongs to $L^1_\Lambda z'$ a.e.; the same proof shows that $\psi_z(z') = f(z, z')$ belongs to $L^1_{\Lambda'} z$ a.e.

Now if $(\gamma, \gamma') \notin \Lambda \times \Lambda'$, we have for instance $\gamma \notin \Lambda$, and thus

$$\hat{f}(\gamma, \gamma') = \int_{G'} \gamma'(-z') \left[\int_G \gamma(-z) \phi_{z'}(z) dm(z) \right] dm'(z') = 0$$

since $\phi_{z'} \in L^1_\Lambda z'$ a.e. and therefore the expression inside [ ] is 0 z' a.e.

Let us now prove that the product $\Lambda \times \Lambda'$ of two Shapiro subsets is Shapiro. We already know that $\Lambda \times \Lambda'$ is nicely placed; by Lemma 1.7 it suffices to show that if $(\gamma, \gamma') \in \Lambda \times \Lambda'$ and if $(f_n) \in B_1(L^1_{\Lambda \times \Lambda'})$ converges to 0 in $L(1, \infty)$ then

$\hat{f}_n(\gamma, \gamma')$ tends to 0. If this is not the case, we may assume, by taking a subsequence if necessary, that

$$\lim_{n\to\infty} f_n = 0 \quad \text{a.e.} \quad \text{and} \quad \lim_{n\to\infty} \hat{f}_n(\gamma, \gamma') = \lambda \neq 0. \tag{17}$$

Consider the sequence $g_n(z') = \int_G |f_n(z, z')| \, dm(z)$. We have $\| g_n \|_1 \leqq 1$, and thus by Lemma 1.2 there exists a sequence $(h_n)_{n \geqq 1}$ of convex combinations of the (g_n)'s which converges a.e. Thus

$$z' \text{ a.e.} \quad \exists M_{z'} \quad \text{s.t.} \quad |h_n(z')| \leqq M_{z'} \quad \forall n. \tag{18}$$

Let us define a sequence (f'_n) of convex combinations of the (f_n)'s by using the same convex combinations which defined (h_n). We clearly have $\| f'_n \|_1 \leqq 1$ and by (18),

$$z' \text{ a.e.} \quad \exists M_{z'} \quad \text{s.t.} \int_G |f'_n(z, z')| \, dm(z) \leqq M_{z'} \quad \forall n. \tag{19}$$

The sequence (f'_n) tends a.e. to 0 since (f_n) does. Now let $\Phi_{n,z'}(z) = f'_n(z, z')$. For almost every z', $\phi_{n,z'}$ tends to 0 z a.e.; since Λ is a Shapiro set and $\phi_{n,z'} \in L^1_\Lambda$, (19) implies

$$z' \text{ a.e.} \quad \lim_{n\to\infty} \hat{\phi}_{n,z'}(\gamma) = 0. \tag{20}$$

Now we let

$$\psi_{n,\gamma}(z') = \hat{\phi}_{n,z'}(\gamma).$$

By (20), $(\psi_{n,\gamma})_{n \geqq 1}$ tends to 0 a.e. Moreover, we have

$$\forall n, \quad |\psi_{n,\gamma}(z')| = |\hat{\phi}_{n,z'}(\gamma)| \leqq \int_G |f'_n(z, z')| \, dm(z)$$

and since $\| f'_n \|_1 \leqq 1$,

$$\forall n, \quad \int_{G'} |\psi_{n,\gamma}(z')| \, dm'(z') \leqq 1.$$

But $\psi_{n,\gamma} \in L^1_{\Lambda'}$ and since Λ' is a Shapiro set, one has

$$\lim_{n\to\infty} \hat{\psi}_{n,\gamma}(\gamma') = 0.$$

But clearly $\hat{\psi}_{n,\gamma}(\gamma') = \hat{f}'_n(\gamma, \gamma')$ and this is not compatible with (17) since the (f'_n)'s are convex combinations of the (f_n)'s. □

REMARK. A subset of $\Lambda \times \Lambda'$ is of course not in general the product of subsets of Λ and Λ' and thus the second part of Theorem 2.7 is not a consequence of the first part.

In the next section of this work, we will apply the above techniques for finding concrete examples of Shapiro sets.

3. Examples

Let us mention first that, by the proof of 2.2, the notions we are considering do not depend upon the discrete group Γ of which Λ is a subset. Therefore we may speak of the properties of Λ without making precise the group Γ in which we are working. Our first example is:

EXAMPLE 3.1. N is a Shapiro set.

This is of course a special case of 2.1. For the sake of completeness, let us give a simple proof of this.

We have to prove that any $\Lambda \subset \mathbf{N}$ is nicely placed. The Hardy space $H^1(D)$ is defined by

$$H^1(D) = \left\{ f \in \mathscr{H}(D) \,\middle|\, \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty \right\}$$

where $\mathscr{H}(D)$ denotes the space of holomorphic functions on the open unit disc D. The Poisson transform $f \to P[f]$ defines an isometry between $L^1_{\mathbf{N}}$ and $H^1(D)$, the inverse being given by $f \to f^*$, where $f^* \in L^1(T)$ denotes the boundary values of f (see [28], Chap. 17). Since $P[f](z) = \Sigma_{n=0}^{\infty} \hat{f}(n) z^n$ for $|z| < 1$, the space $L^1_\Lambda(\mathbf{T})$ corresponds to

$$H^1_\Lambda(D) = \{ f \in H^1(D) \mid f^{(n)}(0) = 0 \text{ for } n \notin \Lambda \}.$$

The Cauchy formula shows that for $r < 1$

$$\sup_{|z| \leqq r} |f(z)| \leqq \| f^* \|_1 (1-r)^{-1}$$

and thus the unit ball of $H^1(D)$ is compact for the topology τ_κ of compact convergence in D. Now, let (f_n^*) be a sequence in the unit ball of L^1_Λ, and $f_n = P[f_n^*]$. We assume that $\lim_{n\to\infty} f_n^* = f$ in $L(1, \infty)$, and we must show $f \in L^1_\Lambda$.

By compactness, there exists a subsequence (f'_n) which converges for τ_κ to $g \in H^1(D)$; it is clear that $g \in H^1_\Lambda(D)$. We let $h_n = f'_n - g$; we have $\lim_{n\to\infty} h_n =$

0 for τ_κ and $\lim_{n\to\infty} h_n^* = f - g^*$ for $L(1,\infty)$. Since $g^* \in L^1_\Lambda$, it is enough to show that $f = g^*$. The metric $L(1,\infty)$ is finer than the metric $L^{1/2}$, and thus

$$\forall \varepsilon > 0, \quad \exists N \text{ s.t. } \forall n, \quad k \geqq N, \quad \int_0^{2\pi} |h_n^*(e^{i\theta}) - h_k^*(e^{i\theta})|^{1/2} d\theta \leqq \varepsilon.$$

The function $|h_n - h_k|^{1/2}$ is subharmonic, and thus

$$\forall n, k \geqq N, \quad \forall r < 1, \quad \int_0^{2\pi} |h_n(re^{i\theta}) - h_k(re^{i\theta})|^{1/2} d\theta \leqq \varepsilon.$$

If k tends to $+\infty$, this shows

$$\forall n \geqq N, \quad \forall r < 1, \quad \int_0^{2\pi} |h_n(re^{i\theta})|^{1/2} d\theta \leqq \varepsilon.$$

Now if r tends to 1, we get

$$\forall n \geqq N, \quad \int_0^{2\pi} |h_n^*(e^{i\theta})|^{1/2} d\theta \leqq \varepsilon$$

and if n tends to infinity, this shows that $\| f - g^* \|_{1/2} \leqq \varepsilon$ for any $\varepsilon > 0$ and thus $f = g^*$. □

REMARKS. (1) Up to the terminology, the above result is proved in [31]; J. Shapiro uses in [31] the fact that $H^1(D) = L^1(T) \cap H^p(D)$ for $p < 1$. Actually, if $L = L^1 \cap X$, where X is a closed subspace of $L(\infty, 1)$, it is clear that Λ is nicely placed; if moreover X^* contains $\mathscr{C}(\hat{\Gamma})$, then Λ is Shapiro.

(2) The above proof shows in particular that $\mathbf{N}$ is nicely placed. Therefore, by Lemma 1.8, we have a proof of the Mooney–Havin result ([23], [14]): $L^1(\mathbf{T})/H^1(D)$ is w.s.c., which uses only the basic properties of $H^1(D)$. Moreover, the conclusion remains true for L^1/L^1_Λ, for any subset Λ of $\mathbf{N}$.

EXAMPLE 3.2. $\mathbf{N}^n$ and $\mathbf{N}^{[\mathbf{N}]}$ are Shapiro sets. $\mathbf{N}^n$ is a Shapiro set by 3.1 and Theorem 2.7. We may also prove it by Corollary 2.2; let $x_1, x_2, \ldots, x_n$ be strictly positive real numbers, independent over $\mathbf{Z}$. We define a one-parameter subgroup of $\mathbf{T}^n$ by

$$\psi(t) = (e^{ix_\kappa t})_{1 \leqq \kappa \leqq n}.$$

We have $\psi^*((n_k)) = \Sigma_{k=1}^n n_k x_k$; it is clear that ψ^* is injective and that $\psi^*(\mathbf{N}^n)$ satisfies (*).

Along the same lines, let x_0 be a transcendental number, with $x_0 > 1$. We define a one parameter subgroup of $\mathbf{T}^{[\mathbf{N}]}$ by

$$\psi(t) = (e^{ix_0^\kappa t})_{1 \leqq \kappa}.$$

It is easily seen that 2.2 shows that $\mathbf{N}^{[\mathbf{N}]}$ is Shapiro.

REMARK. We may as before indentify $L^1_{\mathbf{N}^n}(\mathbf{T}^n)$ with $H^1(D^n)$; therefore 3.2 shows that the space $L^1(\mathbf{T}^n)/H^1(D^n)$ is w.s.c. Actually, one can prove by similar methods that $L^1(\partial U)/H^1(U)$ is w.s.c. for U a symmetric Cartan domain of $\mathbf{C}^n$ or U strictly pseudo-convex with a $\mathscr{C}^2$-smooth boundary ([12], Cor. 21).

EXAMPLE 3.3. Bochner sets are Shapiro sets.

It is proved in [5] that if Λ is a subset of $\mathbf{Z}^2$ which is contained in a cone with opening less than π, then Λ is a Riesz set. It is geometrically clear that we can find $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ such that, if $\psi^* : \mathbf{Z}^2 \to \mathbf{R}$ is defined by

$$\psi^*(n_1, n_2) = n_1 + \alpha n_2$$

then $\psi^*(\Lambda)$ satisfies $(*)$. Therefore, by 2.2, Λ is a Shapiro set.

The extension of the above to subsets of $\mathbf{Z}^n$ — or even $\mathbf{Z}^{[\mathbf{N}]}$ — which are contained in "sharp" cones of $\mathbf{R}^n$ is straightforward. Let us note that analogue proofs work for more general classes of sets; it is easily seen for instance that

$$\Lambda = \{(n, m) \in \mathbf{Z}^2 \mid n^3 \geqq m^2\}$$

is a Shapiro subset of $\mathbf{Z}^2$.

REMARK. Theorem 2.1, together with the Hahn–Banach theorem in $\mathbf{R}^n$, shows that we identify $\mathbf{Z}^n$ with a subset of $\mathbf{R}^n$ and if Λ is a subset of $\mathbf{Z}^n$, then $[\Lambda]$ is contained in the closed convex hull of Λ.

EXAMPLE 3.4. Rudin sets are Shapiro sets.

W. Rudin showed in [27] that if Λ is the union of the $\Lambda(1)$ subset of $\mathbf{Z}$ with $\mathbf{N}$, then Λ is a Riesz set. It is an obvious consequence of Theorem 2.1 that such a set is Shapiro.

REMARK. In [31], J. Shapiro asks the question whether L^1_Λ is relatively closed in L^1 for $\| \ \|_p$ $(p < 1)$ for such a set Λ. I do not know the answer to this question, but at least the $\| \ \|_1$-bounded subsets of L^1_Λ are $\| \ \|_p$-relatively closed in L^1 for $p < 1$; and the present work actually shows that this is enough for obtaining all the desired applications.

EXAMPLE 3.5. Alexandrov sets are Shapiro sets.

A. B. Alexandrov showed in ([1], Appendix) that if Λ is a subset of $\mathbf{Z}^2$ such that

(1) $\{n \in \mathbf{Z} \mid (n \times \mathbf{Z}) \cap \Lambda \neq \varnothing\}$ is bounded from below,
(2) for every $n \in \mathbf{Z}$, the set $(n \times \mathbf{Z}) \cap \Lambda$ is bounded from above or from below in $\mathbf{Z}$,

then Λ is a Riesz set. Let us prove that such a Λ is actually Shapiro.

Since the assumptions (1) and (2) above are hereditary, it is enough to prove that such a Λ is nicely placed. By Lemma 2.5, it is enough to show that if $\alpha = (n, n')$ is not in Λ, then there exists a τ-neighbourhood V_α of α such that $\alpha \notin [V_\alpha \cap \Lambda]$. Let us assume for instance that $(n \times \mathbf{Z}) \cap \Lambda$ is bounded from below. Consider the set $F = (I \times \mathbf{Z}) \cup \{(n, k) \in \Lambda \mid k < n'\}$, where I denotes the set $I = \{l < n \mid (l \times \mathbf{Z}) \cap \Lambda \neq \varnothing\}$.

The sets I and $\{(n, k) \in \Lambda \mid k < n'\}$ are finite by assumption and therefore the set F is τ-closed; we let $V_\alpha = \mathbf{Z}^2 \setminus F$. It is clear that

$$V_\alpha \cap \Lambda \subset C = \{(k, l) \in \mathbf{Z}^2 \mid (n, n'+1) \leqq (k, l)\}$$

where $<$ denotes the lexicographical order on $\mathbf{Z}^2$. By Theorem 2.1 we have $[C] = C$ and thus $[V_\alpha \cap \Lambda] \subset C$ and $\alpha \notin [V_\alpha \cap \Lambda]$.

It is clear that the above proof leads to further examples, in the spirit of Alexandrov's construction.

Arithmetical Examples 3.6. Let us consider several natural subsets of $\mathbf{Z}$, including Y. Meyer's examples of [21]. A useful tool is given by

Lemma 3.6.1. *Let $\mathcal{P}$ be the boolean subalgebra of $P(N)$ generated by the arithmetic progressions. Then every $\Lambda \in \mathcal{P}$ is τ-clopen and thus nicely placed.*

Proof. If Γ is an abelian discrete group and Γ' a sub-group of Γ, recall that

$$G' = \Gamma'^{\perp} = \{g \in G \mid \gamma'(g) = 1 \ \forall \gamma' \in \Gamma'\}.$$

It is easily checked that

$$\hat{m}_{G'} = \mathbf{1}_{\Gamma'}; \quad \widehat{\gamma . m_{G'}} = \mathbf{1}_{\{\gamma\Gamma'\}}. \tag{21}$$

Now if $|\Gamma/\Gamma'| < \infty$, G' is a finite group, and $m_{G'}$ is discrete; therefore (21) shows that the cosets of Γ' are τ-clopen. This remark, together with 2.6(1), proves the lemma. □

The class of nicely placed subsets is stable by intersection; moreover, 3.1 shows that the intervals (bounded or not) of $\mathbf{Z}$ are nicely placed. Thus, Lemma 3.6.1 roughly means that the subsets of $\mathbf{Z}$ which have a simple arithmetic characterization are nicely placed. For instance, we have

LEMMA 3.6.2. *The following sets are τ-closed and thus nicely placed*:

(1) $\Lambda_1 = \{-1\} \cup \mathbf{P}$, *where* $\mathbf{P} = \{p \geqq 1 \mid p \text{ prime number}\}$,

(2) *every subset* Λ_1' *of* Λ_1 *containing* $\{-1, 1\}$,

(3) $S_1 = \{n^2 \mid n \in \mathbf{Z}\}$,

(4) $S_2 = \{a^2 + b^2 \mid (a, b) \in \mathbf{Z}^2\}$,

(5) $(-\mathbf{N}) \cup S_3$, *where* $S_3 = \{a^2 + b^2 + c^2 \mid (a, b, c) \in \mathbf{Z}^3\}$.

PROOF. (1), (2) If $n \notin \{-1, 1\}$, the set $P_n = \{kn \mid k \in \mathbf{Z}\}$ meets Λ_1 in at most one point at $|n|$, thus by 3.6.1

$$n \notin \overline{\Lambda_1' \setminus \{n\}}^{\tau};$$

this proves (2), and (1) is a special case.

(3) If $n < 0$, we consider $P_n = \{n + 3n^2k \mid k \in \mathbf{Z}\}$. We have $P_n \cap S_1 = \varnothing$. Indeed if

$$n + 3n^2k = P^2 = (-n)\cdot(-1 - 3nk)$$

then, since $(-n)$ and $(-1 - 3nk)$ are relatively prime, we must have $(-1 - 3nk) \in S_1$; but the equation $-1 \equiv m^2(3)$ has no solution (this part of the proof is taken from [21]).

If $n \geqq 0$ and $n \in S_1$, there exists a prime number p, and $k \geqq 0$ such that $n = p^{2k+1} \cdot n'$, and p does not divide n'. We consider

$$P_n = \{n + kp^{2k+2} \mid k \in \mathbf{Z}\}. \tag{22}$$

It is clear that every $m \in P_n$ is divisible by p^{2k+1} but not by p^{2k+2} and thus $P_n \cap S_1 = \varnothing$. Again, 3.6.1 concludes the proof.

(4) It is well known that $n \in S_2$ if and only if every prime number $p = 4k + 3$ which divides n has an even power in the decomposition of n (see for instance [30]). This shows that if $n_1 n_2 \in S_2$ and n_1 and n_2 are relatively prime, then n_1 and n_2 belong to S_2.

If $n < 0$, we consider

$$P_n = \{n + 4n^2k \mid k \in \mathbf{Z}\} \quad \text{if } n + 4n^2k = (-n)\cdot(-1 - 4nk) \in S_2,$$

then $(-1 - 4nk) \in S_2$, but this is impossible since $a^2 + b^2 \equiv -1(4)$ has no solutions. If $n \geqq 0$ and $n \in S_2$, there is a prime number $p = 4l + 3$ such that n is divisible by p^{2k+1} but not by p^{2k+2}. If we consider again P_n like in (22), it is clear that $P_n \cap S_2 = \varnothing$, and 3.6.1 concludes the proof.

(5) $n > 0$ does not belong to S_3 if and only if n has the form $n = 4^a(8b - 1)$, for a, b two given integers (see [30], p. 45). For such an n, we let

$$P_n = \{n + 2^{2a+3}k \mid k \in \mathbf{Z}\}.$$

If $l = n + 2^{2a+3}k$ belongs to P_n, one has $l = 4^a(8(b+k) - 1)$, and thus $l \in S_3$; therefore $P_n \cap \{(-N) \cup S_3\} \subset -N$ and $[P_n \cap \{(-N) \cup S_3\}] \subset [-N] = -N$. Now 3.6.1 and Lemma 2.5 conclude the proof. $\square$

I thank B. Host and J. Parreau for having simplified the proof of 3.6.2.

PROPOSITION 3.6.3. *The sets* $(-\mathbf{N}) \cup \mathbf{P}$, $(-\mathbf{N}) \cup S_1$ *and* $(-\mathbf{N}) \cup S_2$ *are Riesz and nicely placed subsets of* $\mathbf{Z}$. *Moreover, any subset* Λ *of* $(-\mathbf{N}) \cup \mathbf{P}$, *and any subset of* $(-\mathbf{N}) \cup S_i$ $(i = 1, 2)$ *containing* S_i $(i = 1, 2)$ *satisfies* L^1/L^1_Λ w.s.c.

PROOF. It is shown in [21] that $(-N) \cup \mathbf{P}$ and $(-N) \cup S_1$ are Riesz; $(-\mathbf{N}) \cup S_2$ is Riesz by 3.6.2 and [21]. Those three sets are nicely placed by 3.6.2 and 2.6.1. If $\Lambda \subset (-\mathbf{N}) \cup \mathbf{P}$, then by 3.6.2 and Lemma 2.5 we have $[\Lambda] \setminus \Lambda \subset \{-1, 1\}$, and 1.8 concludes the proof. The last assertions are consequences of 2.6.2. $\square$

REMARKS 3.6.4. (1) The set $\mathbf{P}$ is not a $\Lambda(1)$-set ([22]). Therefore 3.6.2 and 3.6.3 are not corollaries of Theorem 2.1.

(2) The set $(-\mathbf{N}) \cup S_3$ is not a Riesz set. Indeed it contains the set $\{1 + 4k \mid k \in \mathbf{Z}\}$.

Let us conclude the arithmetic examples by showing that a general result, along the same lines as 3.6.2, can be obtained if we use a stronger arithmetical device:

PROPOSITION 3.6.5. *Let* $Q(n_1, n_2, \ldots, n_k) = \Sigma_{i,j=1}^{n} a_{ij} n_i n_j$ *be a positive-definite quadratic form, with integer coefficients. We assume that for every* $(q_i)_{1 \leq i \leq k}$ *in* $\mathbf{Q}^k$, *there exists* $(n_i)_{1 \leq i \leq k}$ *in* $\mathbf{Z}^k$ *such that* $Q((q_i - n_i)) < 1$. *Then the set* $\Lambda = (-N) \cup Q(\mathbf{Z}^k)$ *is nicely placed.*

PROOF. By assumption, Q satisfies the hypothesis of the lemma of Davenport–Cassels ([30], p. 46); therefore, if $n \in Q(\mathbf{Q}^k)$, then $n \in Q(\mathbf{Z}^k)$.

Let n be in $\mathbf{Z} \setminus \Lambda$; we have $n > 0$ and $n \notin Q(\mathbf{Z}^k)$, and thus $n \notin Q(\mathbf{Q}^k)$. Let us assume that for every prime number p and every $k \geqq 0$, the set

$$P_{p,k} = \{n + mp^k \mid m \in \mathbf{Z}\}$$

meets Λ; this implies that the form

$$Q'(n_1, n_2, \ldots, n_k, n_{k+1}) = Q(n_1, \ldots, n_k) - nn_{k+1}^2$$

represents 0 in every p-adic field $\mathbf{Q}_p$; since $n > 0$, it is clear that Q' represents 0 in $\mathbf{R}$. Now, the Hasse–Minowski theorem (see [30], p. 41) implies that Q represents 0 in $\mathbf{Q}$; if

$$Q'(n_1, n_2, \ldots, n_{k+1}) = 0$$

and

$$(n_1, n_2, \ldots, n_{k+1}) \neq (0, 0, \ldots, 0)$$

we must have $n_{k+1} \neq 0$ and thus $n \in Q(\mathbf{Q}^k)$ which is absurd. Therefore there exists p and k such that $P_{p,k} \cap \Lambda = \varnothing$ and 3.6.1 shows that $n \notin [\Lambda]$. □

Apart from the examples of 3.6.2, the form $Q(n_1, n_2) = n_1^2 + 2n_2^2$, for instance, satisfies the assumptions of 3.6.5.

Example 3.7. Edgar sets are Shapiro sets.

By using the analytic Radon–Nikodym property, G. A. Edgar (personal communication) was able to show that if Λ_i ($i \geqq 1$) is

$$\Lambda_i = \{(n_k) \in \mathbf{Z}^{[N]} \mid n_i > 0,\ n_k = 0 \text{ if } k > i\}$$

and if $\Lambda \subseteq (\bigcup_{i=1}^{+\infty} \Lambda_i)$ satisfies $(\Lambda \cap \Lambda_i)$ finite for every i, then Λ is a Riesz set. Let us show that such a set is Shapiro.

Let $<$ be the "left-to-right" lexicographical order on $\mathbf{Z}^{[\mathbf{N}]}$; that is,

$$(n_k) < (\nu_k) \Leftrightarrow n_{k_0} < \nu_{k_0}$$

if k_0 is the biggest integer for which $n_k \neq \nu_k$. It is clear that if $(n) \in \Lambda_i$ and $(\nu) \in \Lambda_j$ with $i < j$, we have $(n) < (\nu)$. Moreover, for any $(n) \in \mathbf{Z}^{[\mathbf{N}]}$, there exists i_0 such that $(\nu) \in \Lambda_i$ with $i \geqq i_0 \Rightarrow (\nu) > (n)$. Therefore it is clear that any Edgar set Λ satisfies the assumptions of 2.1 and therefore is Shapiro.

We will see below (4.6) that in turn, the fact that those sets are Riesz will imply the convergence of analytic martingales, and thus the two points of view are essentially equivalent.

3.8. Alexandrov's example and its applications. In contrast with the above results, let us present an example, due to A. B. Alexandrov — I want to thank S. V. Kisliakov for having communicated this example to me — of a Riesz subset Λ of $\mathbf{Z}$ which is *not* nicely placed in $\mathbf{Z}$.

For any $j \geqq 0$, we let

$$P_j = \{2^j + k2^{j+1} \mid k \in \mathbf{Z}\}.$$

It is easily seen that $n \in P_j$ if and only if 2^j divides n and 2^{j+1} does not divide n; therefore $\{P_j\}_{j \geq 0}$ is a partition of $\mathbf{Z} \setminus \{0\}$. Now let

$$D_n = \{k2^n \mid |k| \leqq 2^n, k \neq 0\}$$

and

$$\Lambda = \bigcup_{n=0}^{+\infty} D_n.$$

If $m \in D_n$, then 2^n divides m and thus $m \notin P_k$ if $k < n$. Therefore $(\Lambda \cap P_k)$ is contained in $(\bigcup_{n \leqq k} D_n)$; in particular, $(\Lambda \cap P_k)$ is finite for every k.

This observation, together with 3.6.1 shows that if $n \notin \Lambda \cup \{0\}$, there exists V_n such that $n \notin [V_n \cap (\Lambda \setminus \{0\})]$, and then 2.5 shows that $\Lambda \cup \{0\}$ is nicely placed.

We use now the notation of 1.7; if we use a discrete measure ν satisfying (9), it is easily deduced for 2.4 that $n \in \widetilde{(V_n \cap \Lambda)}$ implies $n \in \tilde{\Lambda}$; therefore $\widetilde{\Lambda \cup \{0\}} \supseteq \Lambda$. Now if $\mu \in M_\Lambda$, we have $\mu_s \in M_{\Lambda \cup \{0\}}$ by 1.5 and $\hat{\mu}_s(n) = 0$ for $n \in \Lambda$ by (5) and $\widetilde{\Lambda \cup \{0\}} \supseteq \Lambda$; this clearly shows $\mu_s = 0$ and thus Λ is a Riesz set.

Let us show that Λ is *not* nicely placed. It is easily seen that there exists a sequence $(f_n)_{n \geqq 1}$ of functions in $L^1(\mathrm{T})$ such that $\| f_n \|_1 \leqq 2, \hat{f}_n(0) = 0$ and

$$\lim_{n \to \infty} f_n = \mathbf{1}_{\mathrm{T}} \quad \text{a.e.} \tag{23}$$

By approximation, we may assume that the (f_n)'s are trigonometric polynomials. Using the fact that the functions (z) and (z^{2^n}) have the same distribution, it is easy to deduce from (23) that the function $\frac{1}{2} \cdot \mathbf{1}_{\mathrm{T}}$ is limit in L^0 of a sequence in $B_1(L^1_\Lambda)$; therefore, Λ is not nicely placed, and we actually have $[\Lambda] = \Lambda \cup \{0\}$.

Let us deduce two applications of the above construction. Our next result is an improvement of a result of H. P. Rosenthal [25]. Let us recall that a set Λ is a Rosenthal set if $C_\Lambda(G) = L^\infty_\Lambda(G)$; the Rosenthal sets form a subfamily of the Riesz sets, and H. P. Rosenthal showed [25] that there are such sets which are not $\Lambda(1)$. Actually, one has:

Proposition 3.8.1. *There exists a Rosenthal set which is not nicely placed.*

Proof. A set Λ is said to have a partition for the uniform norm if there exists a partition $(\Lambda_j)_{j \geqq 1}$ of Λ into finite sets, and K such that

$$\sum_{j=1}^{\infty} \| P_j \|_\infty \leqq K \| P \|_\infty \tag{24}$$

for every P, trigonometric polynomial supported by Λ, with $P = \Sigma_{j \geqq 1} P_j$, P_j supported by Λ_j ($j \geqq 1$).

It is easily seen that a set which has a partition for the uniform norm is a Rosenthal set. We let as before

$$D_n = \{k \,.\, 2^n \mid |k| \leqq 2^n, k \neq 0\}.$$

It is easily seen that, for $\varepsilon > 0$ and $n \geqq 1$, there exists n' such that

$$\| P \|_\infty + \| P' \|_\infty \leqq (1+\varepsilon) \| P + P' \|_\infty$$

where P is supported by D_n and P' by $D_{n'}$; it suffices indeed to take n' big enough for P' to be "almost independent" of P as a random variable. Now an easy induction shows that there exists a strictly increasing sequence $(n_k)_{k \geqq 1}$ such that

$$\Lambda = \bigcup_{k=1}^{\infty} D_{n_k}$$

satisfies (24). But the same proof as before shows that $[\Lambda] = \Lambda \cup \{0\}$ and thus Λ is not nicely placed. □

The above examples satisfy $[\Lambda] \setminus \Lambda$ finite, and 1.8 clearly shows that L^1/L^1_Λ is w.s.c. if $[\Lambda] \setminus \Lambda$ is finite. But one has:

Proposition 3.8.2. *There exists a Riesz set Λ in $\mathbf{Z}$ such that $[\Lambda] \setminus \Lambda$ in infinite.*

Proof. We use the same notations as above. For $j \geqq 0$, we let

$$\Lambda_j = \{2^j + k2^{j+1} \mid k \in \Lambda\}$$

and

$$\Lambda' = \{0\} \cup \bigcup_{j=0}^{+\infty} \Lambda_j.$$

A very similar proof shows that $[\Lambda'] = \Lambda' \cup \{2^n \mid n \geqq 0\}$ and that $\tilde{\Lambda}' = \Lambda'$; therefore, if $\mu \in M_{\Lambda'}(T)$, one has $\mu_s \in \mu_{[\Lambda']}$ and $\mu_s = 0$ on $\tilde{\Lambda}' = \Lambda'$; therefore μ_s is supported by the set $\{2^n \mid n \leqq 0\}$ and since this set is Riesz, $\mu_s = 0$ and Λ' is Riesz. □

Let us mention what we proved and used in the proof of 3.8.2 that a set Λ such that $[\Lambda] \setminus \tilde{\Lambda}$ is Riesz is a Riesz set.

It is possible to iterate the above construction for obtaining Riesz sets Λ such that the set $\Lambda' = [\Lambda] \setminus \Lambda$ enjoys a more intricate structure; it is not clear whether the space L^1/L^1_Λ is w.s.c. for such Riesz sets.

4. Applications and related results

A reformulation of F. Lust-Piquard's result [20] is that a subset Λ of Γ is Riesz if and only if the space $\mathscr{C}/\mathscr{C}_{\Gamma\setminus\Lambda}(\hat{\Gamma})$ is an Asplund space.

Our next result asserts that the subclass of Shapiro sets corresponds to a natural subclass of the class of Asplund spaces. Let us recall that E is an M-ideal in its bidual if $\| x^{***} \| = \| x^* \| + \| x^\perp \|$ if $x^{***} = x^* + x^\perp$ in the decomposition $E^{***} = E^* \oplus E^\perp$ of E^{***}.

Proposition 4.1. *Let Λ be a subset of Γ. The following are equivalent*:

(1) *Λ is a Shapiro subset of Γ.*

(2) *$\mathscr{C}(\hat{\Gamma})/\mathscr{C}_{\Gamma\setminus\Lambda}(\hat{\Gamma})$ is an M-ideal in its bidual.*

Proof. (1)$\Rightarrow$(2) If Λ is Shapiro, then $(-\Lambda)$ is nicely placed and thus by 1.3 we have

$$(L^1_{-\Lambda})^{\perp\perp} = L^1_{-\Lambda} \oplus \{(L^1_{-\Lambda})^{\perp\perp} \cap \operatorname{Ker} \pi\} \tag{25}$$

where π is the canonical projection from L^{1**} onto L^1. Since $L^1_{-\Lambda} = \mathscr{C}_{\Gamma\setminus\Lambda}(\hat{\Gamma})^\perp$, it is easily seen that (2) holds if and only if

$$(L^1_{-\Lambda})^{\perp\perp} \cap \operatorname{Ker} \pi = L^1_{-\Lambda}{}^{\perp\perp} \cap \mathscr{C}(\hat{\Gamma})^\perp. \tag{26}$$

If φ belongs to $(L^1_{-\Lambda})^{\perp\perp} \cap \operatorname{Ker} \pi$, then φ may be written

$$\varphi = \lim_U f_n \quad \text{in } (L^{1**}, w^*)$$

with $f_n \in L^1_{-\Lambda}$ and $\| f_n \|_1 \leqq \| \varphi \|_1$ for every n. Since $\varphi \in \operatorname{Ker} \pi$, we have ([12], lemma 7) that $\lim_U f_n = 0$ in $L^0(\mu)$. Since Λ is Shapiro, this implies by 1.7 that $\lim_U \hat{f}_n(\alpha) = 0$ for every $\alpha \in \Lambda$, and $f \in \mathscr{C}(\hat{\Gamma})^\perp$ follows. Therefore

$$(L^1_{-\Lambda})^{\perp\perp} \cap \operatorname{Ker} \pi \subseteq L^1_{-\Lambda}{}^{\perp\perp} \cap \mathscr{C}(\hat{\Gamma})^\perp. \tag{27}$$

But clearly

$$L^1_{-\Lambda} \cap (L^1_{-\Lambda}{}^{\perp\perp} \cap \mathscr{C}(G)^\perp) = \{0\} \tag{28}$$

and the linear algbebra shows that (25), (27) and (28) imply (26).

(2)⇒(1) If $\Lambda' \subset \Lambda$, then $\mathscr{C}_{\Gamma\setminus\Lambda}(\hat{\Gamma}) \subseteq \mathscr{C}_{\Gamma\setminus\Lambda'}(\hat{\Gamma})$ and thus there is a quotient map

$$Q : \mathscr{C}/\mathscr{C}_{\Gamma\setminus\Lambda}(\hat{\Gamma}) \to \mathscr{C}/\mathscr{C}_{\Gamma\setminus\Lambda'}(\hat{\Gamma}).$$

The class of M-ideals in their bidual is stable by quotient maps (see e.g. [13]) and thus (2) is a hereditary property. It is therefore enough to show that (2) implies that Λ is nicely placed. But (2) implies that the space $L^1_{-\Lambda}$ is L-summand of its bidual; by ([12], Lemma 23), this implies that (25) holds, and 1.3 shows that Λ is nicely placed. □

REMARK. It is proved in [19] that $\mathscr{C}(T)/A_0(D)$ is an M-ideal in its bidual; this result is applied in [19] to the obtention of best approximation results. This corresponds to the example $\Lambda = \mathbf{N}$ in the implication (1)⇒(2) of Proposition 4.1. More generally the numerous results (see e.g. [3], [13]) of the M-structure theory may be applied, by 4.1, to Shapiro sets.

PROPOSITION 4.2. *Let Λ be a Shapiro set. If $(f_n)_{n\geqq 1}$ is a sequence in $L^1_\Lambda(G)$ such that*

$$\varphi(g) = \lim_{n\to\infty} \int f_n g\,dm \tag{29}$$

exists for every $g \in \mathscr{C}(G)$, then there exists $f \in L^1_\Lambda$ such that $\varphi(g) = \int fg\,dm$ for every $g \in \mathscr{C}(G)$. Moreover there exists a subsequence (f'_n) such that

$$\lim_{k\to\infty} \frac{1}{k}\left(\sum_{n=1}^{k} f'_n\right) = f \tag{30}$$

a.e. and in $L(1,\infty)$. Moreover if we have $\lim_{n\to\infty} \| f_n \|_1 = \| f \|_1$, then

$$\lim_{n\to\infty} \int f_n g\,dm = \int fg\,dm, \quad \forall g \in L^\infty(G). \tag{31}$$

PROOF. The sequence $(f_n)_{n\geqq 1}$ is $\| \ \|_1$-bounded by the uniform boundedness principle, and its limit belongs to L^1 — and thus to L^1_Λ — since Λ is Riesz. By Lemma 1.2 there exists a subsequence (f'_n) which converges a.e. and in $L(1,\infty)$ to some $g \in L^1$; we have $g \in L^1_\Lambda$ since Λ is nicely placed, and $\hat{g}(\alpha) = \hat{f}(\alpha)$ for every $\alpha \in \Lambda$ by 1.7; therefore $g = f$ and (30) holds. Finally, (31) is a consequence of the easy fact (see [12], proof of theorem 6) that if E is an M-ideal in its bidual, then the w^* and weak topologies coincide on the unit sphere of E^*, and of 4.1. □

Let us mention that the example of the subspace of $L^1(\{-1, 1\}^{\mathbf{N}})$ generated by the coordinate functionals shows that we cannot replace in general the weak topology by the strong topology in (31); this is however possible if $\Lambda = \mathbf{N}$ or $\mathbf{N}^n$ ([24], [12], Th. 22). Since $\Lambda = \mathbf{N}^n$ is a Shapiro set, 4.2 shows:

COROLLARY 4.3. *Let $(f_n)_{n \geq 1}$ be a $\| \quad \|_1$-bounded sequence in $H^1(D^l)$. If $(f_n)_{n \geq 1}$ converges pointwise in $(\dot{D})^l$ to f, then $f \in H^1(D^l)$ and there exists a subsequence (f'_n) such that*

$$\lim_{k \to \infty} \frac{1}{k} \left(\sum_{n=1}^{k} f'^*_n \right) = f^*$$

a.e. and in $L(1, \infty)$ on $\mathbf{T}^l$.

Let us mention that it is possible to replace, in 4.3, D^l by a symmetric Cartan domain U (e.g. $U = B_l$) if we replace T^l by the distinguished boundary ∂U of U (e.g. $\partial B_l = S_l$). On the other hand, the above assertion is false — even if $l = 1$ — if we consider harmonic functions instead of analytic ones. The obstruction here is that $|f|^p$ $(p < 1)$ is in general not subharmonic for f a harmonic function.

Proposition 4.1 will permit us to make a link between the Riesz and Shapiro sets and some decompositions of the functions of L^1_Λ.

LEMMA 4.4. *Let Λ be a subset of Γ. Then, if $\mathscr{C}(G)/\mathscr{C}_{\Gamma\setminus\Lambda}(G)$ is isomorphic — resp. isometric — to a subspace of the space $K(L^2)$ of compact operators on the Hilbert space, then Λ is Riesz — resp. Shapiro.*

PROOF. We let $X_\Lambda(G) = \mathscr{C}(G)/\mathscr{C}_{\Gamma\setminus\Lambda}(G)$. If X_Λ is a subspace of $K(L^2)$ then $X^*_\Lambda = M_{-\Lambda}(G)$ has the Radon–Nikodym property and thus $L^1_{-\Lambda}(G)$ has R.N.P.; now [20] shows that $(-\Lambda)$ and therefore Λ is Riesz.

The space $K(L^2)$ is an M-ideal in its bidual, and this class is hereditary. If X_Λ is isometric to a subspace of $K(L^2)$, X_Λ is an M-ideal in X^{**}_Λ, and 4.1 concludes the proof. □

We will use this lemma with the Hankel operators:

PROPOSITION 4.5. *Let S be a sub-semigroup of the abelian discrete group Γ, such that for every $\alpha \in S$, the equation $\alpha = \alpha_1 + \alpha_2 (\alpha_1, \alpha_2 \in S)$ has only a finite number of solutions. Then:*

(1) *If every $g \in L^1_S$ may be written $g = \Sigma_{i=1}^n h^i_1 h^i_2$ with $h^i_1, h^i_2 \in L^2_S$ and*

(32) $$C\,\|g\|_1 \geqq \sum_{i=1}^{n} \|h_1^i\|_2\,\|h_2^i\|_2$$

then S is a Riesz set.

(2) *If $C = 1$ in (32), S is a Shapiro set.*

PROOF. For $f \in \mathscr{C}(G)$ and $g \in L^2(G)$, we let

$$T_f(g) = \pi_2(fg)$$

where π_2 denotes the orthogonal projection from $L^2_S(G)$ onto $L^2_{-S}(G)$. We consider T_f as an operator from $L^2_S(G)$ to $L^2_{-S}(G)$.

For every $\alpha \in \Gamma$, we have $(\alpha + S) \cap (-S)$ finite, and thus $\mathrm{rk}(T_\alpha) < \infty$. The $\|\ \|_\infty$-denisty of the trigonometric polynomials shows that $T_f \in K(L^2)$ for every $f \in \mathscr{C}(G)$. Moreover it is easily seen that $T_f = 0 \Leftrightarrow f \in \mathscr{C}_{\Gamma\setminus(-S)}(G)$. We let

$$\Phi(f) = \|\dot{f}\|\ \mathscr{C}/\mathscr{C}_{\Gamma\setminus(-S)}(G).$$

We will compare $\Phi(f)$ and $\|T_f\|$. We have

(33) $$\Phi(f) = \sup\left\{\int fg\,dm \;\middle|\; g \in L^1_S,\ \|g\|_1 \leqq 1\right\}.$$

Indeed the unit ball of L^1_S is w^*-dense in the unit ball of $M_S(G) = \mathscr{C}^{\perp}_{\Gamma\setminus(-S)}$ (convolution with "peak" functions). On the other hand

$$\|T_f\| = \sup\left\{\left|\int \pi_2(fg)\bar{h}\,dm\right| \;\middle|\; g \in L^2_S,\ h \in L^2_{-S},\ \|g\|_2 \leqq 1,\ \|h\|_2 \leqq 1\right\},$$

but since $h \in L^2_{-S}$, $\bar{h} \in L^2_S$ and we get

$$\begin{aligned}\int \pi_2(fg)\bar{h}\,dm &= \sum_{\alpha\in\Gamma} \pi_2(fg)(-\alpha)\hat{\bar{h}}(\alpha)\\ &= \sum_{\alpha\in S} \pi_2(fg)(-\alpha)\hat{\bar{h}}(\alpha)\\ &= \sum_{\alpha\in S} fg(-\alpha)\,.\,\hat{\bar{h}}(\alpha)\\ &= \int fg\bar{h}\,dm\end{aligned}$$

and this proves

(34) $$\|T_f\| = \sup\left\{\int fg\bar{h}\,dm \;\middle|\; g \in L^2_S,\ h \in L^2_{-S},\ \|g\|_2 \leqq 1,\ \|h\|_2 \leqq 1\right\}.$$

Now if $g \in L_S^2$ and $h \in L_{-S}^2$, $g\bar{h} \in L_S^1$ since S is a semi-group. Thus, (33) and (34) prove that $\| T_f \| \leqq \Phi(f)$ for every $f \in \mathscr{C}(G)$. Conversely, let f be in $\mathscr{C}(G)$ and g in the unit ball of L_S^1 such that

$$\int fg\,dm \geqq \Phi(f) - \varepsilon.$$

If we write g like in (32), we get

$$\int fg\,dm = \sum_{i=1}^{n} \int fh_1^i h_2^i\,dm \leqq \sum_{i=1}^{n} \| T_f \| \cdot \| h_1^i \|_2 \cdot \| h_2^i \|_2 \leqq C \| T_f \|$$

and this proves $\Phi(f) \leqq C \cdot \| T_f \|$.

We have proved that (32) shows that $\mathscr{C}/\mathscr{C}_{\Gamma \setminus (-S)}$ is C-isomorphic to a subspace of $K(L^2)$; now, 4.4 finishes the proof. □

Examples. It is well-known that $\mathbf{N}$ satisfies (32) with $C = 1$. We have therefore an alternative proof that $\mathbf{N}$ is a Shapiro set. If $n > 1$, it is not known if a decomposition of the form (32) is always possible for $f \in H^1(D^n)$. (I want to thank S. V. Kisliakov for useful information on this matter.) *A fortiori*, if we consider the example

$$S = \{(n, m) \in \mathbf{Z}^2 \mid n^3 \geqq m^2\}$$

of 3.3, nothing seems to be known about the decomposition of the form (32) for $f \in L_S^1$.

I am indebted to G. A. Edgar for this connection between the above results and the analytic martingales considered in [7] and [8], which will conclude this work.

Our last result is an improvement of ([8], Cor. 4.3). Let us recall that an analytic martingale with values in a Banach X is a sequence $X_N(w)$ defined for $w \in T^N$, of the form

$$X_N(w) = \sum_{n=1}^{N} f_n(w_1, w_2, \ldots, w_{n-1}, w_n) \tag{35}$$

where f_n is an integrable function with values in X, which is almost surely in $H_0^1(X)$ of the last variable w_n. With this terminology, one has:

Proposition 4.6. *Let $(X_N)_{N \geqq 1}$ be an analytic martingale with values in L^1. If (X_N) is $\| \;\; \|_1$-bounded, then it converges a.e. and in L^1.*

Proof. It is proved in [11] that L^1 has the analytic Radon–Nikodym property; a straightforward adaptation of the proof shows that if Λ is a Riesz subset of a discrete group Γ, then the bounded martingales of functions in $L^1_\Lambda(L^1)$ converge a.e. and in L^1. Now, if (X_N) has the form (35), we may assume — approximation by trigonometric polynomials — that $\hat{f}_n$ has finite support. Then $X_N \in L^1_\Lambda(L^1)$, where Λ satisfies the assumptions of 3.7; such a Λ is Shapiro and thus it is Riesz, and this concludes the proof. $\square$

Let me mention that it is possible to modify the proof of ([8], Cor. 4.3) given in [8] in order to obtain 4.6 (W. B. Davis, personal communication). Let us conclude this work with

Remark 4.7. It has been shown very recently [32] that if Γ is a countable infinite discrete abelian group, then the family R of Riesz subsets of Γ is a co-analytic non-Borel subset of the compact space $P(\Gamma)$. This result shows in particular that there is no hope of obtaining positive characterizations of the Riesz sets.

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